Relativistic orbits of classical charged bodies in a spherically symmetric electrostatic field.

Hartmut Frommert

E-Mail: spider at seds.org

Dept. of Physics, University of Constance

P.O.Box 55 60 M 678, D-78434 Konstanz, Germany

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Abstract

The orbits of a relativistic charged body in a static, spherically symmetric electrical field are calculated and classified in the classical theory. Contrary to the non-relativistic problem, we find that there is a limiting minimal value for the angular momentum, L_c. Should the actual angular momentum of a charged test body be lower than this limit, the test particle will spiral into the central point charge instead of having (preceding) Keplerian orbits.

Introduction

Within Dirac's theory, or Sommerfeld's semiclassical theory for the fine-structure of the hydrogene atom it is well known that the ground state will become unstable for nuclear charges (of hydrogene-like ions) $Z > 1/\alpha$, see e.g. Greiner 1981. It is the aim of this paper to show, that this phenomenon is not a typical quantum mechanical one, but that also in the classical theory of a relativistic charged point particle within a static Coulomb field there exist non-stable orbits connected with the existence of a critical angular momentum L_c; should the actual angular momentum of the point particle be lower than L_c, the particle will spiral into the central point charge independently from its energy (unstable orbit), while the non-relativistic treatment yields Keplerian orbits in any case (ellipses, parabola and hyperbola). As in the quantum mechanical case mentioned above we neglect the radiative reaction force on the charged point particle.

1 Lagrange function and the equation of motion

Neglecting the radiative back reaction, the Lagrange function of a relativistic point particle (rest mass m_0, charge e) in an underlying electromagnetic potential A_μ is given by

$$\mathcal{L} = -m_0 c^2 \sqrt{\eta_{\mu\nu} v^\mu v^\nu} - \frac{e}{c} v^\mu A_\mu \ , \tag{1}$$

(v^μ: timelike 4-velocity, $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$). In the following, the magnetic potential $A_i, i = 1 \ldots 3$ is assumed to vanish, and $A_0 = c\Phi$ where Φ is the electrostatic potential and a function of the radial coordinate r only. Then the Lagrange function (1) simplifies to

$$\mathcal{L} = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} - e\Phi(r) \ . \tag{2}$$

where v is the absolute value of the 3-velocity. Because of the spherical symmetry of Φ there exists angular momentum conservation, and consequently the motion of the particle will take place in a plane orthogonal with respect to the angular momentum vector; we *choose* as this plane the x–y-plane.

Using plane polar coordinates r and φ, v^2 simplifies to

$$v^2 = \dot{r}^2 + r^2\dot{\varphi}^2 \ , \tag{3}$$

and we can derive the Euler-Lagrange equations for $r(t)$ and $\varphi(t)$. In case of φ we have, since $\frac{\partial \mathcal{L}}{\partial \varphi} \equiv 0$, a conserved angular momentum L, i.e.

$$L = \frac{m_0 r^2 \dot{\varphi}}{\sqrt{1 - \frac{v^2}{c^2}}} = const \ . \tag{4}$$

Inserting v^2 from (3) and resolving for $\dot{\varphi}$ we get:

$$\dot{\varphi} = \frac{c}{r}\sqrt{\frac{1 - \dot{r}^2/c^2}{1 + \left(\frac{m_0 c r}{L}\right)^2}} \tag{5}$$

Since $\mathcal{L}$ is not explicitely dependent on t we have, in addition, energy conservation, i.e. Hamilton's function is a constant; this results in:

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} + e\Phi = const \tag{6}$$

Inserting (3) and resolving for $\dot{r}$ yields

$$\dot{r} = c\sqrt{1 - \left(\frac{m_0 c^2}{E - e\Phi}\right)^2 - \left(\frac{r\dot{\varphi}}{c}\right)^2} \ . \tag{7}$$

Eliminating $\dot{\varphi}$ by eq. (5) one obtains after a short calculation:

$$\dot{r} = c\sqrt{1 - \left(\frac{m_0 c^2}{E - e\Phi}\right)^2 \left(1 + \left(\frac{L}{m_0 c r}\right)^2\right)} \ . \tag{8}$$

Herewith equation (5) takes the form:

$$\dot{\varphi} = \frac{Lc^2}{(E - e\Phi)\, r^2} \tag{9}$$

By combination of (7) and (9) we get the differential equation for calculating the orbit, i.e. $r(\varphi)$:

$$\frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = r^2 \frac{m_0 c}{L} \sqrt{\left(\frac{E - e\Phi}{m_0 c^2}\right)^2 - 1 - \left(\frac{L}{m_0 c r}\right)^2} \tag{10}$$

Substituting $r = 1/s$, this may be rewritten as

$$\frac{ds}{d\varphi} = -\frac{m_0 c}{L} \sqrt{\left(\frac{E - e\Phi}{m_0 c^2}\right)^2 - 1 - \left(\frac{L}{m_0 c}\right)^2 s^2}\ . \tag{11}$$

For the Coulomb potential $\Phi = Q/r = Qs$ we get finally:

$$\frac{ds}{d\varphi} = -\frac{m_0 c}{L} \sqrt{\left[\left(\frac{E}{m_0 c^2}\right)^2 - 1\right] - 2\frac{E}{m_0 c^2}\frac{eQ}{m_0 c^2} s - \left[\left(\frac{L}{m_0 c}\right)^2 - \left(\frac{eQ}{m_0 c^2}\right)^2\right] s^2} \tag{12}$$

Looking now for a more suggestive form of the orbital differential equation, one can take the square of this equation, differentiate with respect to φ, and divide by $2d\varphi/ds$. Thus one obtains

$$\frac{d^2 s}{d\varphi^2} = -\left[1 - \left(\frac{eQ}{cL}\right)^2\right] (s \pm s_0) \tag{13}$$

with

$$s_0 = \frac{\frac{E}{m_0 c^2}\frac{|eQ|}{m_0 c^2}\left(\frac{m_0 c}{L}\right)^2}{1 - \left(\frac{eQ}{cL}\right)^2} \tag{14}$$

The upper sign in (13) applies if e and Q have same sign and the lower sign otherwise. Depending on the value of L, equation (13) has 3 different types of solutions, namely

1. for $L > L_c$: periodic solutions (trigonometric functions) in φ,

2. for $L = L_c$: a limiting algebraic case,

3. and for $L < L_c$: nonperiodic (hyperbolic and exponential functions),

where the "critical" angular momentum L_c is given by

$$L_c = \left|\frac{eQ}{c}\right| \ . \tag{15}$$

For easier processing, it appears appropriate to substitute three new constants for E, L, and Q (or L_c):

$$u := \frac{E}{m_0 c^2} \ , \quad l := \frac{L}{m_0 c} \ , \quad l_c := \frac{L_c}{m_0 c} = \frac{|eQ|}{m_0 c^2} \ , \tag{16}$$

where u is a dimensionless "specific energy", while the "specific angular momentum" l and the "specific critical angular momentum" l_c have the dimension of a length. The length l_c gives the distance at which the absolute value of the electric potential energy, eQ/r, of the test particle (i.e., between the charges e and Q) gets equal to te rest energy $m_0 c^2$ of the particle. This demonstrates the special-relativistic nature of the effects where l_c or L_c plays a role. Then eq. (12) reads:

$$\frac{ds}{d\varphi} = -\frac{1}{l}\sqrt{(u^2-1) \mp 2ul_c s - \left(l^2 - {l_c}^2\right) s^2} \tag{17}$$

The constant s_0, eq. (14) takes the value

$$s_0 = \frac{u l_c}{l^2 - {l_c}^2} \ . \tag{18}$$

2 Integration of the equation of motion

2.1 Large angular momentum $L > L_c$: Keplerian orbits

In case $l > l_c$ or $L > L_c = |eQ|/c$ we can rewrite equation (17) in the following form using (18):

$$\frac{ds}{d\varphi} = -\frac{1}{l}\sqrt{(u^2-1) - \left(l^2 - {l_c}^2\right)\left(s^2 \pm 2s_0 s\right)} \tag{19}$$

After quadratic completion we obtain

$$\begin{aligned}\frac{ds}{d\varphi} &= -\frac{1}{l}\sqrt{\left[(u^2-1)+\left(l^2-{l_c}^2\right){s_0}^2\right]-\left(l^2-{l_c}^2\right)(s\pm s_0)^2} \\ &= -\frac{1}{l}\sqrt{\left[\frac{u^2l^2}{l^2-{l_c}^2}-1\right]\cdot\left[1-\frac{\left(l^2-{l_c}^2\right)^2(s\pm s_0)^2}{(u^2-1)\,l^2+{l_c}^2}\right]}\end{aligned} \tag{20}$$

With respect to the radicand we can substitute

$$s = \mp s_0 + \frac{\sqrt{(u^2-1)\,l^2+{l_c}^2}}{l^2-{l_c}^2}\cos\alpha \tag{21}$$

which yields from (20):

$$\frac{d\alpha}{d\varphi} = \sqrt{1-\left(\frac{l_c}{l}\right)^2} \quad\Longrightarrow\quad \alpha = \sqrt{1-\left(\frac{l_c}{l}\right)^2}\,(\varphi-\varphi_0)\ . \tag{22}$$

Combining (21) and (22) we obtain for $r(\varphi)$:

$$r(\varphi) = \frac{r_0}{\mp 1+\epsilon\cos\left[\sqrt{1-\left(\frac{l_c}{l}\right)^2}\,(\varphi-\varphi_0)\right]} \tag{23}$$

where

$$\epsilon = \frac{\sqrt{(u^2-1)\,l^2+{l_c}^2}}{ul_c} = \sqrt{\frac{1}{u^2}+\left(\frac{l}{l_c}\right)^2-\left(\frac{l}{ul_c}\right)^2} \tag{24}$$

is the numerical excentricity. The upper sign applies in the repulsive and the lower one in the attractive case.

The solution (23) represents a *preceding* Keplerian orbit, i.e. a preceding ellipse, parabola, or hyperbola, depending on the value of $\epsilon < 1, = 1, > 1$ respectively, where in case of repulsive force $\epsilon > 1$ must hold because of $r \geq 0$. In case of bound states it is a periodic orbit with period $2\pi/\sqrt{1-\left(\frac{l_c}{l}\right)^2} > 2\pi$; thus we have a progressive periapsis shift of

$$\delta\varphi = 2\pi\left(\frac{1}{\sqrt{1-\left(\frac{l_c}{l}\right)^2}}-1\right) \tag{25}$$

per cycle caused by the critical value of l_c. For large values of L or l, this expression can be approximated by

$$\delta\varphi \approx \pi \left(\frac{l_c}{l}\right)^2 = \pi \left(\frac{|eQ|}{Lc}\right)^2 \tag{26}$$

In case of scattering states $\epsilon > 1$, we have preceding hyperbola, which means that the asymptotes, calculated from (23), are given by

$$\varphi - \varphi_0 = \pm \frac{1}{\sqrt{1-\left(\frac{l_c}{l}\right)^2}} \arccos\left(\mp\frac{1}{\epsilon}\right) \tag{27}$$

This preceding hyperbola does not coincide with an exact hyperbola, with another excentricity. Instead, its asymptotes, together with its apsid line, precede progressively while the particle moves around the center, at exactly the same rate per passed angle as for the ellipses in the bound above, so that from approach to escape the trajectory has preceded by a total of

$$\delta\varphi = \left(\frac{1}{\sqrt{1-\left(\frac{l_c}{l}\right)^2}} - 1\right) \cdot 2\arccos\left(\mp\frac{1}{\epsilon}\right) \tag{28}$$

If the specific angular momentum l comes close to the limit l_c, the angle $\delta\varphi$ will become larger and larger, so that the particle may orbit the central charge one or more times, before it escapes again to infinity (see Fig. 1).

Such types of orbits are also known from general relativity, where both massive and massless test bodies with small angular momentum have similar trajectories within Schwarzschild's metrical gravitational field (see e.g. Laue 1965 and Misner, Thorne, Wheeler 1973). However, as stated above our results are a purely special relativistiv effect, while in the Schwarzschild case the non-linear structure of the gravitational field has an additional impact.

2.2 The limiting case $L = L_c$: Quadratic spiral trajectories

For $L = L_c$, i.e. $l = l_c$ (corresponding to $s_0 = \infty$) eq. (17) takes the form

$$\frac{ds}{d\varphi} = \sqrt{\frac{2u}{l_c}\,(s_1 \mp s)} \tag{29}$$

with

$$s_1 = \frac{u^2 - 1}{2ul_c} \tag{30}$$

(the upper sign is valid for repulsive forces, the lower one for the attractive case). The solution reads

$$s = \frac{u}{2l_c}\,(\varphi - \varphi_0)^2 \pm s_1 \ ; \tag{31}$$

according to (30), s_1 is positive, 0, or negative as $u > 1$, $= 1$, or < 1 respectively. Then the trajectory is given by

$$r = \mp \frac{r_0}{a\,(\varphi - \varphi_0)^2 \mp 1} \tag{32}$$

where

$$r_0 = \frac{2ul_c}{|u^2 - 1|} \ , \quad a = \frac{u^2}{|u^2 - 1|}$$

Since r must be positive, it is valid for

- $u < 1$ and attractive forces: The trajectory has a maximal distance r_0 from the origin at $\varphi = \varphi_0$ and spirals quadratically into the origin as $|\varphi - \varphi_0| \longrightarrow \infty$.

- $u > 1$ and attractive forces:

$$|\varphi - \varphi_0| > 1/\sqrt{a} = \frac{\sqrt{u^2 - 1}}{u} \ . \tag{33}$$

  The trajectory comes from infinity at $|\varphi - \varphi_0| = 1/\sqrt{a}$, and spirals quadratically into the origin for $|\varphi - \varphi_0| \longrightarrow \infty$.

- $u > 1$ and repulsive forces:

$$|\varphi - \varphi_0| < 1/\sqrt{a} = \frac{\sqrt{u^2-1}}{u} \; . \tag{34}$$

In this case, $r_0 = r(\varphi = \varphi_0)$ represents the minimal distance from the central charge and for $|\varphi - \varphi_0| \longrightarrow 1/\sqrt{a}$ the trajectory runs out to infinity without and spiralling.

No solution exists in the repulsive case for $u < 1$. For $u = 1$ ($s_1 = 0$), we have only a solution in the attractive case, namely

$$s = (u/2l_c)(\varphi - \varphi_0)^2 \; , \tag{35}$$

which comes from infinity at $\varphi = \varphi_0$, and spirals quadratically into the origin with $|\varphi - \varphi_0| \longrightarrow \infty$.

2.3 Small angular momentum $L < L_c$: Exponential spiral orbits

In the last case $L < L_c$, i.e. $l < l_c$, we can rewrite eq. (17) as

$$\left(\frac{ds}{d\varphi}\right)^2 l^2 = \left[1 + \frac{u^2 l^2}{{l_c}^2 - l^2}\right] \left\{ \frac{(l_c^2 - l^2)^2}{(u^2-1)\, l^2 + {l_c}^2} \,(s \mp s_0)^2 - 1 \right\} \tag{36}$$

(s_0 according to (18) and upper sign for repulsive, the lower one for attractive forces). In the following, it is convenient to introduce the abbrevation

$$b = \sqrt{\left(\frac{l_c}{l}\right)^2 - 1} \tag{37}$$

which is a positive real constant. We discuss the solutions of this equation of motion for the *attractive* case ($eQ < 0$) first. Then we have to distinguish three cases, corresponding to the value of the "specific energy" constant u:

a. $u < 1$, i.e. sum of kinetic and potential energy negative:

In this case the solution of (36) reads (see Fig. 2)

$$r = \frac{r_0}{1 + a\left\{\cosh\left[b\left(\varphi - \varphi_0\right)\right] - 1\right\}} \tag{38}$$

where

$$r_0 = \frac{{l_c}^2 - l^2}{\sqrt{{l_c}^2 - \left(1 - u^2\right) l^2} - u l_c} > 0 \tag{39}$$

$$a = \frac{1}{1 - \dfrac{u l_c}{\sqrt{{l_c}^2 - \left(1 - u^2\right) l^2}}} > 1\ . \tag{40}$$

The trajectory described by equation (38) has its greatest distance r_0 from the origin at $\varphi = \varphi_0$, and spirals exponentially into the central charge for both $\varphi \longrightarrow \infty$ and $\varphi \longrightarrow -\infty$, as

$$r \longrightarrow \frac{2r_0}{a} e^{-b|\varphi - \varphi_0|}\ . \tag{41}$$

b. $u = 1$, i.e. sum of kinetic and potential energy zero:

The solution of (36) takes the form (Fig. 3)

$$r = \frac{r_1}{\cosh\left[b\left(\varphi - \varphi_0\right)\right] - 1} \tag{42}$$

with

$$r_1 = \frac{{l_c}^2 - l^2}{l_c} > 0 \tag{43}$$

The trajectory approaches infinity for $\varphi = \varphi_0$ and spirals exponentially into the origin for $\varphi \longrightarrow \infty$ as

$$r \longrightarrow 2r_1 e^{-b|\varphi - \varphi_0|}\ . \tag{44}$$

c. $u > 1$, i.e. sum of kinetic and potential energy positive:

Equation (36) has the solution (Fig. 4)

$$r = \frac{r_0}{a\left\{\cosh\left[b\left(\varphi - \varphi_0\right)\right] - 1\right\} - 1} \tag{45}$$

with

$$r_0 = \frac{{l_c}^2 - l^2}{ul_c - \sqrt{{l_c}^2 - (1-u^2)\, l^2}} > 0 \tag{46}$$

$$a = \frac{1}{\dfrac{ul_c}{\sqrt{{l_c}^2 - (1-u^2)\, l^2}} - 1} > 0 \tag{47}$$

Because the distance r must be always positive, the range of φ is restricted so that the denominator in (45) remains positive:

$$|\varphi - \varphi_0| > \frac{1}{b}\text{arcosh}\left(1 + \frac{1}{a}\right) = \varphi_\infty \tag{48}$$

The trajectory does not enter the φ interval $\varphi_0 - \varphi_\infty < \varphi < \varphi_0 + \varphi_\infty$. It comes from infinity at

$$\varphi = \varphi_0 \pm \varphi_\infty$$

and spirals exponentially into the origin for $|\varphi - \varphi_0| \longrightarrow \infty$ as

$$r \longrightarrow \frac{2r_0}{a} e^{-b|\varphi - \varphi_0|} , \tag{49}$$

i.e. exactly as in the first case.

In a second step we discuss the motion for the repulsive case. For this, the solution of (36) reads

$$r = \frac{r_0}{1 - a\left\{\cosh\left[b\left(\varphi - \varphi_0\right)\right] - 1\right\}} \tag{50}$$

with

$$r_0 = \frac{{l_c}^2 - l^2}{ul_c - \sqrt{{l_c}^2 - (1-u^2)\, l^2}} \tag{51}$$

$$a = \frac{1}{\dfrac{ul_c}{\sqrt{{l_c}^2 - (1-u^2)\, l^2}} - 1} . \tag{52}$$

Because of $r > 0$ it follows that $u > 1$ must hold. The trajectory has a *minimal* distance $r_0 = r(\varphi_0)$ and goes to infinity without any spiralling for

$$|\varphi - \varphi_0| \longrightarrow \varphi_\infty = \frac{1}{b}\mathrm{arcosh}\left(\frac{u l_c}{\sqrt{{l_c}^2 - (1-u^2)\, l^2}}\right) \;;$$

it never leaves the φ interval $\varphi_0 - \varphi_\infty < \varphi < \varphi_0 + \varphi_\infty$. It comes from the infinity at $\varphi = \varphi_0 - \varphi_\infty$, has its closest approach, $r = r_0$, at $\varphi = \varphi_0$, and leaves again to infinity at $\varphi = \varphi_0 - \varphi_\infty$.

3 Summary

The different classes of orbits discussed above for the special relativistic Coulomb problem are summarized in the following tables:

Attractive case:

	$E < m_0c^2$ bound states	$E = m_0c^2$	$E > m_0c^2$ scattering states
$L > L_c$	preceding ellipse eq. (23), $0 \leq \epsilon < 1$	preceding parabola eq. (23), $\epsilon = 1$	preceding hyperbola eq. (23), $\epsilon > 1$
$L = L_c$	quadratical spiral from maximal r_0 into the center, eq. (32), lower signs	quadratic spiral from infinity into the center, eq. (35)	quadratic spiral from infinity into the center, eq. (32), "+" before RHS, "−" in denominator
$L < L_c$	exponential spiral from maximal r_0 into the center, eq. (38), Fig. 2	exponential spiral from infinity into the center, eq. (42), Fig. 3	exponential spiral from infinity into the center, eq. (45), Fig. 4

Repulsive case:

	$E > m_0c^2$ scattering states
$L > L_c$	preceding repulsive hyperbola eq. (23), upper sign, $\epsilon > 1$
$L = L_c$	quadratic approach to and escape from minimal r_0 to infinity eq. (32), upper signs
$L < L_c$	exponential approach to and escape from minimal r_0 to infinity eq. (50)

Obviously, in classical special-relativistic electrostatics, there exists a limiting angular momentum: If a charged body which is attracted by the electric field has less angular momentum than this critical value, it will spiral into the source, and this already without taking into account the radiative energy losses. The physical meaning is that low angular momentum test charges get so close to the central charge, i.e. into such a strong field, that they are accelerated to relativistic velocities and can then no more escape.

Besides that this result is of interest on its own, one may look for applications, which can be expected in that part of physics where special relativity plays a role, while quantum effects stay weak. However with respect to applications the radiative back reaction will be important and must be taken into account. This will be done in a subsequent paper. Nevertheless we will give an estimation of the critical situation discussed above: For electrons as test particles, it is necessary to concentrate a charge of

$$Q = l_c \cdot \frac{m_0 c^2}{e} \approx 4.2 \cdot 10^{12} e \cdot (l_c/cm)$$

in a volume of a radius smaller than l_c, i.e. for a 1-cm radius, more than $4 \cdot 10^{12}$ elementary charges had to be stabilized and localized in this volume, generating a voltage at its surface which corresponds to the electron's $m_0\, c^2/e$, i.e. more than $5.11 \cdot 10^5\ V$; "classical" test bodies would require even a much higher central charge. It may be difficult to realize such a densely packed charge. However it is the hope that the radiative reaction force will improve the experimental conditions.

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Figure Captions:

Fig. 1 The test particle may orbit the central charge one (**a**) or more times (**b**), before it escapes again to infinity, if the angular momentum L approaches L_c.

Fig. 2 Orbits for $L < L_c$, $u < 1$ (bound states)

Fig. 3 Orbits for $L < L_c$, $u = 1$ (limiting case)

Fig. 4 Orbits for $L < L_c$, $u > 1$ (scattering states)